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RANK–ONE PERTURBATIONS AND TRANSFORMATIONS OF CENTROSYMMETRIC MATRICES

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Abstract. We study the eigen structure of rank–one perturbations of centrosymmetric matrices and reduce the eigen/inverse/determinant problem of rank–one perturbations of centrosymmetric matrices to corresponding problems half the size of the original ones. We also study transformations of centrosymmetric matrices.

1. Introduction

Centrosymmetric matrices have a rich eigenstructure that has been studied extensively in the literature (see [**17, 14, 1, 11, 2, 5, 3**]). Many results for centrosymmetric matrices have been generalized to wider classes of matrices that arise in a number of applications (see [**7, 8, 13**]). Perturbations of matrices, especially rank–one perturbations, were studied by many (see [**16, 10, 15, 4**]). In particular, many people studied rank–one perturbations of real symmetric matrices (by studying rank–one perturbations of diagonal matrices). A clear relationship between the eigenvalues and the eigenvectors of the perturbated matrix and the eigenvalues and the eigenvectors of the original matrix has not been found yet. We will prove that if H is an $n \times n$ centrosymmetric matrix, w is an $n \times 1$ vector, and u is an $n \times 1$ symmetric or skew–symmetric vector, then H and $H + wu^T$ share some eigenvalues and some eigenvectors. If H is nondefective, then H and $H + wu^T$ share $\lfloor n/2 \rfloor$ eigenvalues and $\lfloor n/2 \rfloor$ linearly independent eigenvectors when u is symmetric, and $\lceil n/2 \rceil$ eigenvalues and $\lceil n/2 \rceil$ linearly independent eigenvectors when u is skew–symmetric. When $w = u$, we will reduce the computation of the remaining eigenvalues and the remaining eigenvectors of $H + uu^T$ significantly. In addition, we will reduce the determinant/inverse problem of $H + uu^T$ to two smaller determinant/inverse problems. We will prove that if H is a centrosymmetric matrix and if the number of linearly independent eigenvectors of H is γ, then H and JH share γ linearly independent eigenvectors. We will also mention the relationship between the eigenvalues of H and the eigenvalues of JH.

2. Preliminaries

We employ the following notation. We denote the transpose of a matrix A by A^T and the determinant of A by $\det(A)$. We use the notation $\lceil x \rceil$ for the smallest integer greater than or equal to x and $\lfloor x \rfloor$ for the largest integer less than or equal to x. As usual, I denotes the identity matrix.

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By the *main counterdiagonal* (or simply *counterdiagonal*) of a square matrix we mean the positions which proceed diagonally from the last entry in the first row to the first entry in the last row.

Definition 2.1. The *counteridentity* matrix, denoted J, is the square matrix whose elements are all equal to zero except those on the counterdiagonal, which are all equal to 1.

We note that multiplying a matrix A by J from the left results in reversing the rows of A and multiplying A by J from the right results in reversing the columns of A.

A vector x is called *symmetric* if $Jx = x$ and *skew–symmetric* if $Jx = -x$. Throughout this paper let $\mathcal{E}$ be the set of all $n \times 1$ vectors that are either symmetric or skew–symmetric.

Definition 2.2. A matrix A is *centrosymmetric* if $JAJ = A$.

Centrosymmetric matrices have applications in many fields like communication theory, harmonic differential quadrature, statistics, differential equations, numerical analysis, engineering, physics, and pattern recognition. Rank–one perturbations of centrosymmetric matrices have applications in several fields. For applications of centrosymmetric matrices, see [**12, 11, 9, 6**]. Note that symmetric Toeplitz matrices are symmetric centrosymmetric.

We use the following known result (see [**2, 5**], for example).

Theorem 2.3. *Let H be an $n \times n$ centrosymmetric matrix, let $\delta = \frac{n}{2}$, and let $\xi = \frac{n-1}{2}$. Then*

(i) *If n is even, then H can be written as*

$$H = \begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix},$$

where A, J and C are $\delta \times \delta$ matrices. If n is odd, then H can be written as

$$\begin{bmatrix} A & x & JCJ \\ y^T & q & y^TJ \\ C & Jx & JAJ \end{bmatrix},$$

where A, J and C are $\xi \times \xi$ matrices, x and y are $\xi \times 1$ vectors, and q is a scalar.

(ii) *If n is even, then H is similar to*

$$\begin{bmatrix} A - JC & 0 \\ 0 & A + JC \end{bmatrix}.$$

If n is odd, then H is similar to

$$\begin{bmatrix} A - JC & 0 & 0 \\ 0 & q & \sqrt{2}y^T \\ 0 & \sqrt{2}x & A + JC \end{bmatrix}.$$

(iii) *If n is even, then the eigenvalues of H are the eigenvalues of $F_1 := A - JC$ and the eigenvalues of $G_1 := A + JC$. Then the eigenvectors corresponding to the eigenvalues of F_1 can be chosen to be skew–symmetric of the form $(u^T, -u^T J)^T$, where u is an eigenvector of F_1, while the eigenvectors corresponding to the eigenvalues of G_1 can be chosen to be symmetric of the form $(u^T, u^T J)^T$, where u is an eigenvector of G_1. If n is odd, then the eigenvalues of H are the eigenvalues of F_1 and the eigenvalues of*

$$G_2 := \begin{bmatrix} q & \sqrt{2}y^T \\ \sqrt{2}x & A + JC \end{bmatrix}.$$

Then the eigenvectors corresponding to the eigenvalues of F_1 can be chosen to be skew–symmetric of the form $(u^T, 0, -u^T J)^T$, where u is an eigenvector of F_1, while the eigenvectors corresponding to the eigenvalues of G_2 can be chosen to be symmetric of the form $(u^T, \sqrt{2}\alpha, u^T J)^T$, where $(\alpha, u^T)^T$ is an eigenvector of G_2.

3. Our Results

In this section, we prove several properties of rank–one perturbations of centrosymmetric matrices. We also study the effect of reversing the rows/columns of a centrosymmetric matrix on its eigenvalues and eigenvectors.

Proposition 3.1. *Let u be an $n \times 1$* symmetric *vector, let w be an $n \times 1$ vector, let H be an $n \times n$ nondefective centrosymmetric matrix, and let $M = H + wu^T$. Then, H and M share at least $\lfloor n/2 \rfloor$ eigenvalues and $\lfloor n/2 \rfloor$ linearly independent eigenvectors.*

Proof. By Theorem 2.3, $\lfloor n/2 \rfloor$ linearly independent eigenvectors of a nondefective centrosymmetric matrix can be chosen to be skew–symmetric. Now if z is an $n \times 1$ skew–symmetric vector, then $u^T z = 0$. Hence, if z is skew–symmetric, then (λ, z) is an eigenpair of H if and only if (λ, z) is an eigenpair of M. □

Lemma 3.2. *Let $\delta = \frac{n}{2}$ and let $\xi = \frac{n-1}{2}$.*

(i) *Let n be even, let u be an $n \times 1$ symmetric/skew–symmetric vector, and let $S = uu^T$. Then u and S can be written as*

$$u = \begin{bmatrix} v \\ p \end{bmatrix}$$

and

$$S = \begin{bmatrix} R & LJ \\ JL & JRJ \end{bmatrix},$$

where v is $\delta \times 1$, J is $\delta \times \delta$, $R = vv^T$, $p = Jv$ if u is symmetric and $p = -Jv$ if u is skew–symmetric, and $L = R$ if u is symmetric and $L = -R$ if u is skew–symmetric.

(ii) *Let n be odd, let u be an $n \times 1$ symmetric/skew–symmetric vector and let $S = uu^T$. Then u and S can be written as*

$$u = \begin{bmatrix} v \\ \zeta \\ p \end{bmatrix},$$

and

$$S = \begin{bmatrix} R & \zeta v & LJ \\ \zeta v^T & \zeta^2 & \zeta v^T J \\ JL & \zeta Jv & JRJ \end{bmatrix},$$

where v is $\xi \times 1$, J is $\xi \times \xi$, $R = vv^T$, $\zeta = k$ (some scalar k) if u is symmetric and $\zeta = 0$ if u is skew–symmetric, $p = Jv$ if u is symmetric and $p = -Jv$ if u is skew–symmetric, and $L = R$ if u is symmetric and $L = -R$ if u is skew–symmetric.

Proof. It suffices to prove the case when u is symmetric and n is even. The rest of the proof is similar. The form of u follows from the definition of symmetric vectors, and the remaining results follow immediately from the fact that, since $(Jv)^T = v^T J$,

$$\begin{bmatrix} v \\ Jv \end{bmatrix} \begin{bmatrix} v^T & (Jv)^T \end{bmatrix} = \begin{bmatrix} vv^T & vv^T J \\ Jvv^T & Jvv^T J \end{bmatrix}.$$

□

Now we use Theorem 2.3 to reduce the eigen problem of M (when $w = u$) to two smaller eigen problems. One of these reduced eigen problems will be the same as one of the reduced eigen problems of H. In other words, one of the reduced eigen problems of M will be free of u.

Theorem 3.3. *Let u be an $n \times 1$* symmetric *vector, let H be an $n \times n$ centrosymmetric matrix, let $M = uu^T + H$, let H be decomposed as in Theorem 2.3, let R be as in the previous lemma, and let $\delta = \frac{n}{2}$.*

(i) *If n is even, then the eigenvalues of M are the eigenvalues of $F_1 := A - JC$ and the eigenvalues of $G_3 := A + JC + 2R$, and the eigenvectors of M can be determined from the eigenvectors of F_1 and the eigenvectors of G_3. Moreover, the shared eigenvalues between H and M are the eigenvalues of F_1 and the shared eigenvectors are the eigenvectors determined from the eigenvectors of F_1. If, in addition, M is nondefective, then δ eigenvalues and δ skew–symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_1 f_i = \lambda_i f_i,$$

and δ eigenvalues and δ symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$G_3 g_i = \mu_i g_i.$$

(ii) *If n is odd, then the eigenvalues of M are the eigenvalues of F_1 and the eigenvalues of G_4, and the eigenvectors of M can be determined from the eigenvectors of F_1 and the eigenvectors of G_4, where*

$$G_4 = \begin{bmatrix} q+k^2 & \sqrt{2}(y+kv)^T \\ \sqrt{2}(x+kv) & A+JC+2R \end{bmatrix}.$$

Moreover, the shared eigenvalues between H and M are the eigenvalues of F_1 and the shared eigenvectors are the eigenvectors determined from the eigenvectors of F_1. If, in addition, M is nondefective, then $\lfloor n/2 \rfloor$ eigenvalues and $\lfloor n/2 \rfloor$ skew–symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$F_1 f_i = \lambda_i f_i,$$

and $\lceil n/2 \rceil$ eigenvalues and $\lceil n/2 \rceil$ symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$G_4 g_i = \mu_i g_i.$$

Proof. We will prove only Part (i). The proof of Part (ii) is similar. From Theorem 2.3, H can be written as

$$H = \begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix},$$

where A, J and C are $\delta \times \delta$. From the previous lemma, uu^T can be written as

$$uu^T = \begin{bmatrix} R & RJ \\ JR & JRJ \end{bmatrix},$$

where J and R are $\delta \times \delta$. Thus, M can be written as

$$M = \begin{bmatrix} A+R & J(C+JR)J \\ C+JR & J(A+R)J \end{bmatrix}.$$

Now apply Theorem 2.3 to M. □

Note that the previous theorem provides another proof for Proposition 3.1. Note also that every eigenspace of M has a basis consisting of vectors from $\mathcal{E}$.

In the following corollary, we reduce the inverse/determinant problem of nonsingular rank–one perturbations of centrosymmetric matrices to a smaller inverse/determinant problem. Both I and J in the corollary are $n/2 \times n/2$.

Corollary 3.4. *With the same notation as the previous theorem*

(i) *If n is even and M is nonsingular, then*

$$\det(M) = \det(A - JC) \cdot \det(A + JC + 2R),$$

and

$$M^{-1} = \frac{1}{2}\begin{bmatrix} \alpha^{-1}+\beta^{-1} & (\beta^{-1}-\alpha^{-1})J \\ J(\beta^{-1}-\alpha^{-1}) & J(\alpha^{-1}+\beta^{-1})J \end{bmatrix},$$

where $\alpha = A - JC$ and $\beta = A + JC + 2R$.

(ii) *If n is odd and M is nonsingular, then*

$$\det(M) = \det(A - JC) \cdot \det\left((q+k^2)(A + JC + 2R) - 2(x+kv)(y+kv)^T\right),$$

and if $q + k^2 = 1$, then

$$M^{-1} = \begin{bmatrix} \frac{1}{2}(N^{-1}+\alpha^{-1}) & -N^{-1}(x+kv) & \frac{1}{2}(N^{-1}-\alpha^{-1})J \\ -(y+kv)^T N^{-1} & 1 + 2(y+kv)^T N^{-1}(x+kv) & -(y+kv)^T N^{-1} J \\ \frac{1}{2}J(N^{-1}-\alpha^{-1}) & -JN^{-1}(x+kv) & \frac{1}{2}J(N^{-1}+\alpha^{-1})J \end{bmatrix},$$

where $\alpha = A - JC$ and $N = A + JC + 2R - 2(x+kv)(y+kv)^T$.

Proof. It suffices to prove Part (i).

$$QMQ^T = \begin{bmatrix} A - JC & 0 \\ 0 & A + JC + 2R \end{bmatrix}$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ I & J \end{bmatrix}.$$

Thus,

$$M^{-1} = Q^T \begin{bmatrix} (A-JC)^{-1} & 0 \\ 0 & (A+JC+2R)^{-1} \end{bmatrix} Q.$$

□

Note that if n is odd, M is nonsingular, and $q+k^2 \neq 1$, then M^{-1} can be obtained using the formula for the case when $q + k^2 = 1$, and it can also be obtained directly using a proof similar to the proof of Part (i) of the previous corollary. Note also that if H is nonsingular and n is even, then

$$\begin{aligned} \frac{\det(M)}{\det(H)} &= \frac{\det(A + JC + 2R)}{\det(A + JC)} \\ &= \frac{\det(A + JC) \cdot \det\left(I + (A + JC)^{-1}(2R)\right)}{\det(A + JC)} \\ &= 1 + 2v^T (A + JC)^{-1} v. \end{aligned}$$

If n is odd, then $\det(M) = \det(H) \cdot \frac{N_1}{N_2}$, where

$$N_1 = \det\left((q+k^2)(A + JC + 2R) - 2(x+kv)(y+kv)^T\right)$$

and

$$N_2 = \det\left(q(A + JC) - 2xy^T\right).$$

Now we derive similar results to those in the previous proposition, theorem, and corollary, for the case when u is skew–symmetric instead of symmetric. The proofs are similar to the previous ones.

Proposition 3.5. *Let u be an $n\times 1$* skew–symmetric *vector, let w be an $n\times 1$ vector, let H be an $n\times n$ nondefective centrosymmetric matrix, and let $M = H + wu^T$. Then, H and M share at least $\lceil n/2\rceil$ eigenvalues and $\lceil n/2\rceil$ linearly independent eigenvectors.*

Theorem 3.6. *Let u be an $n\times 1$* skew-symmetric *vector, let H be an $n\times n$ centrosymmetric matrix, let $M = uu^T + H$, let H be decomposed as in Theorem 2.3, let R be as in Lemma 3.2, and let $\delta = \frac{n}{2}$.*

(i) *If n is even, then the eigenvalues of M are the eigenvalues of $F_2 := A - JC + 2R$ and the eigenvalues of $G_1 := A + JC$, and the eigenvectors of M can be determined from the eigenvectors of F_2 and the eigenvectors of G_1. Moreover, the shared eigenvalues between H and M are the eigenvalues of G_1 and the shared eigenvectors are the eigenvectors determined from the eigenvectors of G_1. If, in addition, M is nondefective, then δ eigenvalues and δ skew–symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_2 f_i = \lambda_i f_i,$$

and δ eigenvalues and δ symmetric orthonormal eigenvectors of M can be determined from solving the equation

$$G_1 g_i = \mu_i g_i.$$

(ii) *If n is odd, then the eigenvalues of M are the eigenvalues of F_2 and the eigenvalues of G_2, and the eigenvectors of M can be determined from the eigenvectors of F_2 and the eigenvectors of G_2, where*

$$G_2 = \begin{bmatrix} q & \sqrt{2}y^T \\ \sqrt{2}x & A + JC \end{bmatrix}.$$

Moreover, the shared eigenvalues between H and M are the eigenvalues of G_2 and the shared eigenvectors are the eigenvectors determined from the eigenvectors of G_2. If, in addition, M is nondefective, then $\lfloor n/2\rfloor$ eigenvalues and $\lfloor n/2\rfloor$ skew–symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$F_2 f_i = \lambda_i f_i,$$

and $\lceil n/2\rceil$ eigenvalues and $\lceil n/2\rceil$ symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$G_2 g_i = \mu_i g_i.$$

Note that the previous theorem provides another proof for Proposition 3.5. Note also that every eigenspace of M has a basis consisting of vectors from $\mathcal{E}$.

Corollary 3.7. *With the same notation as the previous theorem*

(i) *If n is even and M is nonsingular, then*

$$\det(M) = \det(A - JC + 2R)\cdot\det(A + JC),$$

and

$$M^{-1} = \frac{1}{2}\begin{bmatrix} \alpha^{-1}+\beta^{-1} & (\beta^{-1}-\alpha^{-1})J \\ J(\beta^{-1}-\alpha^{-1}) & J(\alpha^{-1}+\beta^{-1})J \end{bmatrix},$$

where $\alpha = A - JC + 2R$ *and* $\beta = A + JC$.

(ii) *If* n *is odd and* M *is nonsingular, then*

$$\det(M) = \det(A - JC + 2R)\cdot\det\left(q(A+JC) - 2xy^T\right),$$

and if $q = 1$, *then*

$$M^{-1} = \begin{bmatrix} \frac{1}{2}(N^{-1}+\alpha^{-1}) & -N^{-1}x & \frac{1}{2}(N^{-1}-\alpha^{-1})J \\ -y^TN^{-1} & 1+2y^TN^{-1}x & -y^TN^{-1}J \\ \frac{1}{2}J(N^{-1}-\alpha^{-1}) & -JN^{-1}x & \frac{1}{2}J(N^{-1}+\alpha^{-1})J \end{bmatrix},$$

where $\alpha = A - JC + 2R$ *and* $N = A + JC - 2xy^T$.

Note that if n is odd, M is nonsingular, and $q \neq 1$, then M^{-1} can be obtained using the formula for the case when $q = 1$, and it can also be obtained directly using a proof similar to the proof of Part (i) of Corollary 3.4.

With the same notation as Theorem 3.6, note that if H is nonsingular and n is even, then

$$\begin{aligned} \frac{\det(M)}{\det(H)} &= \frac{\det(A-JC+2R)}{\det(A-JC)} \\ &= \frac{\det(A-JC)\cdot\det(I+(A-JC)^{-1}(2R))}{\det(A-JC)} \\ &= 1 + 2v^T(A-JC)^{-1}v. \end{aligned}$$

If n is odd, then

$$\begin{aligned} \frac{\det(M)}{\det(H)} &= \frac{\det(A-JC+2R)}{\det(A-JC)} \\ &= 1 + 2v^T(A-JC)^{-1}v. \end{aligned}$$

Now we study transformations of centrosymmetric matrices. In particular, we study the effect of reversing the rows/columns of a centrosymmetric matrix on its eigenvalues and its eigenvectors.

Proposition 3.8. *The transformation* L *defined by* $L(M) = JM$ *is a bijection on centrosymmetric matrices.*

Theorem 3.9. *Let* H *be an* $n \times n$ *centrosymmetric matrix and let* γ *be the number of linearly independent eigenvectors of* H. *Then,* H *and* JH *share* γ *linearly independent eigenvectors, and* λ *is an eigenvalue of* H *if and only if* λ *or* $-\lambda$ *is an eigenvalue of* JH.

Proof. γ linearly independent eigenvectors of H can be chosen to be symmetric or skew–symmetric. Since JH is also centrosymmetric, then the same thing holds for JH. Now it is easy to prove that if z is symmetric, then (λ, z) is an eigenpair of H if and only if (λ, z) is an eigenpair of JH. Also, it is easy to prove that if

z is skew–symmetric, then (λ, z) is an eigenpair of H if and only if $(-\lambda, z)$ is an eigenpair of JH. □

The previous theorem also follows from the proof of Theorem 2.3 and it is important because it reveals a relationship between the eigenvalues of some Toeplitz matrices and the eigenvalues of some Hankel matrices. Note that a matrix M is Toeplitz if and only if JM is Hankel and vice versa.

Corollary 3.10. *Let H be an $n \times n$ nondefective centrosymmetric matrix. Then $\lceil n/2 \rceil$ eigenvalues of JH are the same as $\lceil n/2 \rceil$ eigenvalues of H and the remaining eigenvalues of JH are the negatives of the remaining eigenvalues of H. Moreover, H and JH share n linearly independent eigenvectors.*

Proof. Let $m = \lfloor n/2 \rfloor$. By Theorem 2.3, we can determine m linearly independent skew–symmetric eigenvectors of H and $n-m$ linearly independent symmetric eigenvectors. Since JH is also centrosymmetric, then the same facts apply to JH. □

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