Studying The Linear Complex Pt Symmetric Potential

R.J. Lombard[1], R. Mezhoud[2]

[1] Groupe de Physique théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex- France,

roland.lombard@laposte.net

[2] Faculté des Sciences, Université de Boumerdes- Algeria

Abstract

We investigate the spectrum of the linear complex PT symmetric potential $V(x) = \lambda|x| + icx$. Semi analytical solutions are given by using properties of the Airy functions. The numerical integration of the differential equation system is discussed. We show that the number of eigenstates with a real eigenvalue is limited, depending on the ratio c/λ and on the quantum number n, reflecting a spontaneous breaking of the PT symmetry. For the ground state ($n = 0$), we conjecture the eigenvalue to be real whatever the value of c.

Keywords: *Airy function, Complex symmetric potential.*

Introduction

The discovery of complex PT symmetric potentials, with real eigenvalues, has generated a large amount of works ([1, 2]). Analytical solutions have been given, often belonging to real potentials admitting an analytical solution. The archetype is the harmonic oscillator given by Znojil [3]. Other examples have been reported ([4, 5, 6, 7, 8, 9, 10]).

To our knowledge, the linear potential case has been only partially investigated by Bender et al [11]. The present work discusses the spectrum of

$$v(x) = \lambda|x| + icx \tag{1}$$

The Airy function

In the D = 1 space, on the right half plan ($x \geq 0$), the Schrödinger equation reads

$$\left(-\frac{d^2}{dx^2} + (\lambda + ic)x\right)\Psi_n^+(x) = E_n\Psi_n^+(x) \tag{2}$$

Similarly, for $x \leq 0$, we have:

$$\left(-\frac{d^2}{dx^2} - \lambda x \_ icx\right)\Psi_n^-(x) = E_n\Psi_n^-(x) \tag{3}$$

By a trivial change of variables, both equations take the same form with complex conjugate coupling constant.

Let us consider the positive x half line. By making the usual changes

$$g = \lambda + ic \quad , \quad z = g^{1/3}\left(x - \frac{E_n}{g}\right) \tag{4}$$

Eq(2) is transformed into the well-known Airy function differential equation [12]

$$\left(\frac{d^2}{dz^2} - z\right)\Psi_n^+(x) = 0 \tag{5}$$

Because we are looking for square integrable solutions, we retain

$$\Psi_n^+(z) \propto Ai(z) \tag{6}$$

The second linear independent solution, Bi(z), is divergent as $\Re z \to \infty$, and is thus eliminated.

From the above argument, and the analyticity properties of the Airy functions, we have

$$\Psi_n^-(z) = \overline{\Psi}_n^+(z) \quad \text{and} \quad \Psi_n(z) = \Psi_n^-(z) \cup \Psi_n^+(z) \; ; \quad z = z(x, E_n) \tag{7}$$

The solutions, and in particular the eigenvalues, are fixed by the continuity condition at x = 0. Actually, if we write

$$\Psi_n(x) = U_n(x) + iV_n(x) \tag{8}$$

the boundary conditions are given by (up to a normalisation factor)

$$U_n(0) = 1 \quad U_n'(0) = 0 \quad V_n(0) = 0 \quad V_n'(0) \neq 0 \qquad \text{n even} \tag{9}$$

and

$$U_n(0) = 0 \quad U_n'(0) \neq 0 \quad V_n(0) = 1 \quad V_n'(0) = 0 \qquad \text{n odd} \tag{10}$$

Here, the U ′ and V ′ denote the derivatives with respect to x.

With these boundary conditions, the Eq. (5) is solved for $x \geq 0$. The full solutions are then obtained by adding the symmetric or anti-symmetric partners for $x \leq 0$.

The procedure to find the solutions follows the one of the real case. However, with complex arguments the boundary conditions at $x = 0$ are not satisfied automatically simultaneously for the real and the imaginary parts.

The trick is to remark that the eigenfunctions are determined up to a constant arbitrary phase. Usually, this phase is irrelevant. Here, however, this degree of freedom can be used to match the continuity conditions at $x = 0$.

Working on the right half plane, we introduce

$$\Psi_+(z) \rightarrow \Psi_+(z)e^{i\theta_0(n)} = Ai(z)e^{i\theta_0(n)} \tag{11}$$

Thus, we search pairs E_n, $\theta_0(n)$ such that the Airy function satisfies the boundary conditions. However, the system has not necessarily solvable. The absence of solution signals the eigenvalues to be actually complex, and the symmetry to be spontaneously broken. The occurrence of such a situation clearly depends on the ratio $|c|/\lambda$ and on the quantum number n. This result agrees with the fact that the potential ix has no purely real eigenvalue, as shown by Bender and Boettcher [1].

Because we rely on numerical determinations, it is not possible to fix the limits between the real and the complex spectrum without tedious numerical works. For this reason, here we shall merely discussed this aspect on the basis of selected examples.

For the sake of illustration, a few eigenvalues have been calculated. For the fixed value $\lambda = 1$, we choose c = 0.1, c = 0.5 and c = 1.0 as typical examples. The results are displayed in table 1 and compared to the c = 0 case. For c = 0.5 and c = 1.0, the occurrence of complex eigenvalues starts at n = 3 and n = 1, respectively. In each of these last cases, two complex eigenvalues have been determined above the last real one. Note that the results do not depend on the sign of c. They are actually complex conjugate of each other. Consequently, we just quote results for positive c values.

The results displayed in table 1 suggest the ground state eigenvalue to be real even at large c. For this reason, searches have been pushed up to c = 200. The corresponding E_0 are indeed real. The evolution of $E_0(c)$ is displayed in fig 1. It is well fitted by

$$E_0(c) = 1.01879 + (1.5258 \pm 0.0199)c - (0.4335 \pm 0.0093)c^{1.15} \tag{12}$$

The negative coefficient of th $c^{1.15}$ term seems to indicate a saturation, which is confirmed by a logarithmic fit, though less precise on the considered c interval.

$$E_0(c) = 1.01879 + (136.2 \pm 5.8)\log\left[1 + (0.00643 \pm 0.00037)|c|\right] \tag{13}$$

Comparing again these results with those of Bender and Boettcher [1], we recall that for V(§) = $(i§)^N$ these authors noticed the following features. For $N \leq 1.42207$, only the ground state has a real energy. This is translated in our work by the disappearance of real eigenvalues for excited states as c is increasing. Furthermore, as N approaches 1 from above, the ground state energy diverges. This last fact incites us to conjecture that adding a small (eventually infinitesimal) |x| component in the potential transforms this divergence into a logarithmic one.

Integrating the differential equations

Besides the use of the Airy function, we have investigated the numerical integration of the coupled differential equations.

Setting $\Psi_n(x) = U_n(x) + iV_n(x)$, the system to be solved is

$$-U_n'(x) + \lambda|x|U_n(x) \pm icV_n(x) = E_n U_n(x) \tag{14}$$

$$-V_n'(x) + \lambda|x|V_n(x) \pm icU_n(x) = E_n V_n(x)$$

The integration is achieved on the right half plane x ≥ 0, the solutions for x ≤ 0 being the symmetric or anti-symmetric partners of the positive x ones, as stated above. The boundary conditions are given by Eqs (9) and (10) at x = 0. At large distances, the modulus of the wave

function $\sqrt{(U^2(x) + V^2(x))}$ must tend to zero to satisfy the integrability condition.

By choosing E_n and the value of the non-zero derivative at the origin, the system is integrated via a Runge-Kutta procedure. The solution is selected via the behavior of the wave function modulus: it must tangentially approach zero at large distances. As it is well known, applying this criterion is a very difficult task [15, 16].

In practice, solutions are retained up to the point where they reach a minimum close to zero. The wave function modulus with the lowest minimum value is selected as the solution. It clearly means that in this way the eigenvalues and the wave functions are obtained with a limited accuracy.

For the sake of comparison with results obtained from the Airy functions, the system (14) has been integrated for λ = 1 and c = 0.5. The eigenvalues for n = 0, 1 and 2 agree with the ones listed in table 1 within 1 %. The selected solutions have their minimum reaching $10^{-5} - 10^{-6}$ of the modulus value at the origin.

The agreement is of the same order for the wave functions, being close to 0.1 % near the origin. It may reach a factor 2 in the tails, when the modulus reaches $10^{-5} - 10^{-6}$ of its value at x = 0.

That a solution with a real eigenvalue does not exist is more difficult to test with the integration of the differential equation system than we the Airy functions. Actually the basic criterion of an absolute minimum value of the modulus as it approaches zero is not met. Numerous equivalent approximate solutions are found, and the lowest possible eigenvalues violate the strict inequality

$$E_n\left(\pm c \neq 0\right) \geq E_n\left(c = 0\right) \tag{15}$$

Conclusions

We have studied the spectrum of the PT symmetric linear potential $V(x) = \lambda|x| + icx$. Two methods have been applied. First, use is made of the fact that on each half plane the solution of the Schrödinger equation is the complex Airy function Ai(z). The eigenvalues are found by imposing continuity conditions at x = 0. They are determined numerically. According to the value of $|c|/\lambda$, part of the spectrum has real eigenvalues E_n, in accordance with the PT symmetry of the Hamiltonian. However, as c or n increases the eigenvalues become complex, showing a spontaneous breaking of the symmetry.

Special attention has been paid to the ground state. The eigenvalues have been found real up to c = 200/ We conjecture that indeed the ground state has a real eigenvalue up to very large c values. The recourse to numerical methods does not allow us to fix a limit. is comforted by the fact the

The second method consists in the numerical integration of the differential equation system for the real and the imaginary parts of the wave function. Its results are closed to the ones obtained with the Airy function. The arising of complex eigenvalues is marked by the absence of clear criterion to fix a real eigenvalue.

Acknowledgements

We express our thanks to A. Comtet, J. Mares and M. Znojil for valuable comments. Thanks are due also to V. Lafage for numerous discussions concerning the numerical methods. Finally, R. Mezhoud expresses his thanks for the hospitality extended to him by the IPN Orsay.

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Table 1. Real eigenvalues of the linear PT symmetric potential. The eigenvalues do not depend on the sign of c

n	c = 0.0	c = 0.1	c = 0.5	c = 1.0
0	1.0188	1.0245	1.1525	1.4877
1	2.3381	2.3563	2.8320	3.8545 ± i 0.300
2	3.2483	3.2660	3.5274	5.2996 ± i 0.234
3	4.0880	4.1202	4.8147 ± i 0.240	
4	4.8201	4.8459	5.6683 ± i 0.214	
5	5.5206	5.5653		
6	6.1633	6.1947		
7	6.7867	6.8440		
8	7.3722	7.4070		
9	7.9441	8.0140		
10	8.4885	8.5234		

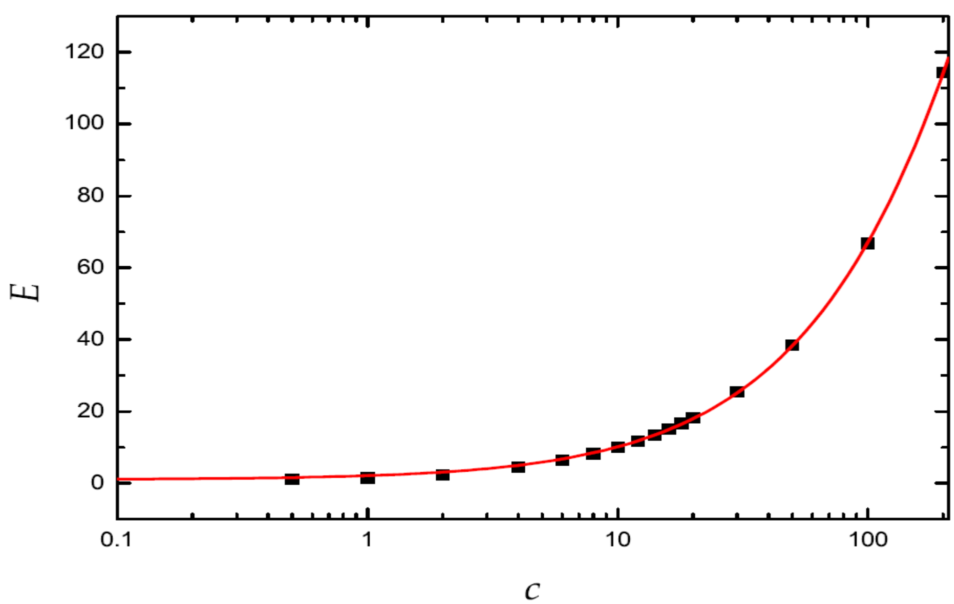

Figure 1. Eigenvalues E0 as function of c. The line gives the fit by Eq. (13)

دراسة الجهد PT التماثلي الخطي المركب

رولاند لومبارد [1] ، مزهود [2]

[1]مجموعة من الفيزياء النظرية ، معهد الفيزياء النووية ، roland.lombard@laposte.net

[2]كلية العلوم ، جامعة بومرداس ، الجزائر

الملخص

في هذه الدراسة نتحقق من طيف الجهد PT التماثلي الخطي المركب. تم طرح حلول شبه تحليلية باستخدام خصائص إقتران ايري. كما تم معالجة المعادلة التفاضلية للنظام بطريقة التكامل العددي. تم إثبات أن عدد الحلول الحقيقية للنظام محدودة وتعتمد على النسبة c/λ والعدد الكمي n مما يعكس انهيار لحظي للتماثل.

نؤكد أن الحل للمستوى الأرضي يكون دائماً حقيقياً لجميع قيم c.

الكلمات المفتاحية: *اقتران ايري, طيف التماثلي الخطي المركب*